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Entropy and Gravitation

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Abstract

Einstein's gravitational theory is analyzed from a thermodynamic point of view. A thermodynamic potential characterizing the sources of gravitational fields is presented. By means of this potential the entropy production density is derived. Einstein's equations with dissipative terms appear as linear phenomenological laws in the sense of irreversible thermodynamics. Some thermodynamic influences on gravitational phenomena are discussed.

1. *Introduction*

In many physically interesting cases the sources of gravitational fields can be treated as thermodynamic systems. In such situations it is not difficult to interpret certain terms of Einstein's theory of gravitation in a thermodynamic way. For instance, the energy-momentum tensor as a part of the field equations represents the thermodynamic energy aspect, since it contains information about stresses, energy flows, and energy densities caused by thermodynamic processes. The statement that the divergence of this tensor vanishes may be interpreted as relativistic formulation of the first law of thermodynamics.

The connection between entropy and gravitational phenomena is not obvious at first sight. However, in nonrelativistic thermodynamics entropy is a very important quantity. Robert Emden (1938) excellently pointed out the meaning of the entropy principle and its relation to the energy principle, when he wrote

> As a student, I read with advantage a small book by F. Wald entitled *The Mistress of the World and her Shadow*. These meant energy and entropy. In the course of advancing knowledge the two seem to me to have exchanged places. In the huge manufactory of natural processes, the principle of entropy occupies the position of manager, for it dictates the manner and method of the whole business, whilst the principle of energy merely does the book-keeping, balancing credits and debits.

Let us check the validity of this assertion within Einstein's theory.

2. *Thermodynamic Description of Gravitational Sources*

To describe gravitational sources we use a deductive method comparable to the method of thermodynamic potentials in ordinary phenomenological thermodynamics. As a special thermodynamic system we choose an imperfect fluid. Generalizations are possible (see Concluding Remarks). The state of the fluid is characterized by the metric tensor g_{ik} and two thermodynamic state variables (tensors of rank 0), temperature T and rest mass density ρ. We denote this set of independent state variables by V_A,

$$(V_A) \equiv (g_{ik}, T, \rho) \tag{2.1}$$

We shall see in the next sections that

$$L = L(V_A, V_{A,i}, V_{A,i,k}) \equiv -[(1/2\kappa_0)R + F]^{1} \tag{2.2}$$

has all properties of a thermodynamic potential for gravitational sources. R and $F = F(T, \rho)$ are the curvature scalar and the density of free energy, respectively. For later calculations we introduce the abbreviations

$$v \equiv \rho^{-1}, \qquad f = f(T, v) \equiv \rho^{-1}F(T, \rho), \qquad s \equiv -\left(\frac{\partial f}{\partial T}\right)_v, \qquad p \equiv -\left(\frac{\partial f}{\partial v}\right)_T,$$

$$\epsilon \equiv \rho(f + Ts), \quad \mu \equiv f + pv \tag{2.3}$$

and call them specific volume, specific free energy, specific entropy, pressure, energy density, and specific free enthalpy (Gibbs function), respectively, in analogy with equivalent definitions in nonrelativistic thermodynamics.

3. *Dynamical Behavior of Gravitational Sources*

Comparing the actual state characterized by V_A with virtual states characterized by

$$\bar{V}_A = V_A + \delta V_A = V_A + (\mathscr{L}_\xi V_A)\delta\omega \tag{3.1}$$

we describe the dynamical behavior of the system by the variational principle

$$\delta \int_{V_4} L \sqrt{g}\, d^4x \geqslant 0 \tag{3.2}$$

$$\delta\omega \geqslant 0, \qquad \delta\omega|_{(V_4)} = 0, \qquad (\delta\omega)_{,i}|_{(V_4)} = 0 \tag{3.3}$$

$\mathscr{L}_\xi$ means Lie derivative with respect to a timelike vector field ξ_k,

$$\xi_k \xi^k < 0 \tag{3.4}$$

The potential L multiplied by the root of the fundamental determinant $g \equiv -\det g_{ik}$ is integrated over a certain domain V_4 in space-time (d^4x is the

[1] $\kappa_0 = 8\pi k c - 4$, where k is the gravitational constant.

four-dimensional volume element). $\delta\omega$ and its derivative $(\delta\omega)_{,i}$ are infinitesimal quantities vanishing at the boundary (V_4) of V_4. Since the Lie derivative with respect to a timelike vector field is a kind of a partial time derivative, the variation consists in a comparison of the actual state with virtual future states here defined by $\delta\omega > 0$. It is an essential statement of the variational principle that a vector field ξ_i connecting actual states with future states always exists. We shall see that ξ_k is closely connected with temperature and four-velocity of the medium. Thus temperature gets a fundamental meaning as a geometrical property in space-time.

Using the boundary conditions (3.3), we obtain

$$\int_{V_4} \frac{\delta L\sqrt{g}}{\delta V_A} \delta V_A \, d^4x \geqslant 0 \quad (3.5)$$

[2]

instead of (3.2) ($\delta/\delta V_A$ means variational derivative). Since the inequality (3.5) must hold for any variation (3.1) with arbitrary values $\delta\omega \geqslant 0$, we must have

$$\frac{\delta L\sqrt{g}}{\delta V_A} \mathscr{L}_\xi V_A \geqslant 0 \quad (3.6)$$

at every point of the domain V_4. We call

$$\sigma = g^{-1/2} \frac{\delta L\sqrt{g}}{\delta V_A} \mathscr{L}_\xi V_A \quad (3.7)$$

entropy production density. Then inequality (3.6) is the second law of thermodynamics. Let us prove this interpretation. First of all let us show that σ forms a four-divergence. The potential L as a tensor of rank 0 remains unchanged under an infinitesimal transformation of coordinates

$$x^{i'} = x^i + \xi^i \delta\tau \quad (3.8)$$

($\delta\tau$ is an infinitesimal constant parameter). In this case Noether's theorem states

$$\frac{\delta L\sqrt{g}}{\delta V_A} \mathscr{L}_\xi V_A = (\sqrt{g}\overset{\text{can}}{S^i})_{,i} \quad (3.9)$$

where

$$\overset{\text{can}}{S^i} = \frac{-1}{\sqrt{g}} \left\{ \mathscr{L}_\xi V_A \left[\frac{\partial L\sqrt{g}}{\partial V_{A,i}} - \left(\frac{\partial L\sqrt{g}}{\partial V_{A,i,k}} \right)_{,k} \right] + (\mathscr{L}_\xi V_A)_{,k} \frac{\partial L\sqrt{g}}{\partial V_{A,i,k}} - \xi^i L\sqrt{g} \right\} \quad (3.10)$$

We call $\overset{\text{can}}{S^i}$ canonical entropy flow density.

[2] Sum convention for A.

Equations (3.9), (3.6), and (3.7) completely correspond with the non-relativistic second law

$$\frac{\partial s}{\partial t} + \operatorname{div} \mathbf{s} = \sigma \geqslant 0 \tag{3.11}$$

which states that the entropy s of a volume element is changed by the entropy flow $\mathbf{s}$ through the surface and by production $(\sigma > 0)$ in the interior. The equality sign in (3.11) and (3.6) characterizes thermodynamic equilibrium. To calculate the left-hand side of (3.6) for the system considered we introduce the unit vector u^i by

$$\xi^i = \xi u^i, \qquad u_i u^i = -1 \tag{3.12}$$

ξ is the norm of ξ^i.

The following considerations will justify the identification[3]

$$\xi = T^{-1} \tag{3.13}$$

Now, putting (2.1) and (2.2) into (3.6), we obtain

$$\frac{1}{2}\left[\frac{1}{\kappa_0}(R^{ik} - \tfrac{1}{2}Rg^{ik}) + (\epsilon + p)u^i u^k + pg^{ik}\right] \mathscr{L}_\xi g_{ik} - (\rho u^i)_{;i}\frac{\mu}{T} \geqslant 0 \tag{3.14}$$

by means of (2.3), (3.12), and (3.13).

The left-hand side of (3.14) has the typical form of an entropy production density $\sigma = \Sigma_M J_M X_M$ expressed as a sum of products of thermodynamic "fluxes"

$$(J_M) = \left(\frac{1}{\kappa_0}[R^{ik} - \tfrac{1}{2}Rg^{ik}] + (\epsilon + p)u^i u^k + pg^{ik}, [\rho u^i]_{;i}\right) \tag{3.15}$$

and "forces"

$$(X_M) = \left(\frac{1}{2}\mathscr{L}_\xi g_{ik}, -\frac{\mu}{T}\right) \tag{3.16}$$

where the "forces" cause the "fluxes."

Indeed, decomposing

$$\mathscr{L}_\xi g_{ik} = \xi_{i;k} + \xi_{k;i} = \left(\frac{u_i}{T}\right)_{;k} + \left(\frac{u_k}{T}\right)_{;i}$$

into parts parallel and perpendicular to u^i (projection tensor: $h_{ik} \equiv g_{ik} + u_i u_k$, $h_{ik}u^k = 0$),

$$h_m{}^i h_n{}^k \mathscr{L}_\xi g_{ik} = T^{-1} h_m{}^i h_n{}^k (u_{i;k} + u_{k;i}) \tag{3.17}$$

$$u^i h_n{}^k \mathscr{L}_\xi g_{ik} = (T^{-1})_{,i} h_n{}^i - T^{-1} u_{n;k} u^k \tag{3.18}$$

$$u^i u^k \mathscr{L}_\xi g_{ik} = -2(T^{-1})_{,k} u^k \tag{3.19}$$

[3] $c = 1$.

and interpreting tentatively u^i as the four-velocity of the medium, we see that $\mathscr{L}_\xi g_{ik}$ contains the velocity gradient ("rate of deformation") (3.17) causing viscosity effects in hydrodynamics and the thermal gradient (3.18) [including Eckart's relativistic addition $T^{-1}\dot{u}_n = T^{-1} u_{n;k}u^k$ (Eckart, 1940)] giving rise to heat conduction. Thus, $\mathscr{L}_\xi g_{ik}$ unites these expressions in a four-dimensional way.

The process connected with (3.19) is usually assumed to be frozen. In the same way as in nonrelativistic thermodynamics we assume linear homogeneous relations between the fluxes and forces,

$$J_M = \sum_N L_{MN} X_N \tag{3.20}$$

especially

$$\frac{1}{\kappa_0}(R^{ik} - \tfrac{1}{2}Rg^{ik}) + (\epsilon + p)u^i u^k + pg^{ik} = L^{iklm}(\tfrac{1}{2}\mathscr{L}_\xi g_{lm}) \tag{3.21}$$

As in general use, we postulate the conservation of rest mass ("frozen equilibrium of rest mass production"),

$$(\rho u^i)_{;i} = 0 \tag{3.22}$$

Equations (3.21) are Einstein's gravitational equations for an imperfect fluid and, together with (3.22), supply the required conditions allowing us to determine the dependence of the state variables V_A on the coordinates x^i (that means the dynamical behavior of the system). The source term of the gravitational field in (3.21) consists of two parts. On the left-hand side we have the energy-momentum tensor of a perfect fluid,

$$\overset{0}{T}{}^{ik} \equiv -(\epsilon + p)u^i u^k - pg^{ik} \tag{3.23}$$

whereas the right-hand side represents the action of the irreversible processes

$$\overset{1}{T}{}^{ik} \equiv \tfrac{1}{2}L^{iklm}\,\mathscr{L}_\xi g_{lm} \tag{3.24}$$

The interpretation of $\overset{0}{T}{}^{ik}$ implies the identification of u^i with the four-velocity of the fluid. Hence, the above interpretation of $\mathscr{L}_\xi g_{ik}$ has been correct and $\overset{1}{T}{}^{ik}$ describes the influence of viscosity and heat conduction on gravitational phenomena. The material tensor L^{iklm} has to fulfil certain symmetry relations (Onsager–Casimir reciprocity relations). Moreover, it is restricted by the condition $\sigma \geqslant 0$. According to the isotropy of the fluid L^{iklm} consists of the material coefficients of viscosity and heat conduction multiplied by vectors u^i and tensors g^{ik}. Because of (3.12) and (3.13), the vector ξ^i is called the temperature vector (see, e.g., Synge, 1957). It represents the relativistic temperature aspect. Let us emphasize that the derivation of Einstein's equations which characterize the dynamical behavior may be compared with the determination of equilibrium conditions in ordinary thermodynamics by means of variational principles for the thermodynamic potentials.

4. *Thermodynamic Properties of Gravitational Sources*

Differentiating the characteristic potential L with respect to the state variables V_A we obtain the thermodynamic properties (state functions) of the system in the same way as in ordinary thermodynamics.

Let us consider the first derivatives only. To ensure covariance we use variational derivatives. The meaning of both $\partial L/\partial\rho = g^{-1/2}\,\delta(Lg^{1/2})/\delta\rho = -\mu$ and $\partial L/\partial T = \rho s$ is clear [cf. equation (2.3)]. $g^{-1/2}\,\delta(Lg^{1/2})/\delta g_{ik}$ cannot be interpreted directly, but

$$S^i = 2\xi_k g^{-1/2}\frac{\delta(Lg^{1/2})}{\delta g_{ik}} = \frac{1}{\kappa_0}\xi_k R^{ik} - \xi^i\left(\frac{R}{2\kappa_0} + F\right) \tag{4.1}$$

is a fundamental quantity. It may be verified

$$S^i = \overset{\text{can}}{S^i} - \frac{1}{2\kappa_0}(\xi^{k;i} - \xi^{i;k})_{;k} \tag{4.2}$$

Therefore

$$S^i{}_{;i} = \sigma \tag{4.3}$$

S^i corresponds to the entropy flow used in nonrelativistic irreversible thermodynamics. Its normal component vanishes at the surface of an adiabatically isolated system. [For proof replace the curvature quantities in (4.1) by the energy-momentum tensor.]

5. *Concluding Remarks*

Considering imperfect fluids we have seen that the entropy principle determines the explicit form of Einstein's equations, which appear as linear phenomenological laws in the sense of irreversible thermodynamics. But the method presented is applicable to other thermodynamic systems, too, for example to relativistic solids $\{L = -[(1/2\kappa_0)R + F(T, \epsilon_{ik})], \epsilon_{ik}$ = deformation tensor$\}$ or fluids (solids) in electromagnetic fields $\{L = -[(1/2\kappa_0)R + F - \frac{1}{4}B^{ik}H_{ik}], B_{ik}, H_{ik}$ = electromagnetic field tensors$\}$. In all cases known the sum $-[(1/2\kappa_0)R + F]$ is a keystone of the characteristic potential completed by Lagrangians of nonmetric fields. The close connection between the characteristic potential L and Lagrangians might stimulate a thermodynamic interpretation of the variational principles of classical field theory in the sense of Section 3.

A detailed description of the dynamical behavior of a system cannot renounce the integration of Einstein's equations (3.21) together with (3.22). Nevertheless helpful statements on interactions between thermodynamic and gravitational properties can be derived by means of integral theorems. Thus it is possible to show that a spatially closed universe with a positive energy density ϵ and a positive pressure p can never be in thermodynamic equilibrium.[4] Thermodynamic

[4] A proof was given by the author in Kramer et al. (1972).

equilibrium is defined by vanishing of the "forces" X_M of all irreversible processes, i.e. (Ehlers, 1961)[5],

$$\sigma = 0, \qquad \mathscr{L}_\xi g_{ik} = \xi_{i;k} + \xi_{k;i} = 0$$

A second example shows the close connection between entropy S, gravitational mass M, and characteristic potential L. Let us assume that a system without entropy exchange with the surroundings has two equilibrium states geometrically characterized by hypersurfaces $x^4 = \text{const}$ ($\xi^i = \delta_4{}^i$). At the intermediate states irreversible processes take place. Integrating the fourth component of (4.1) over the whole volume of the system and subtracting the earlier values from the later ones we obtain ($\xi^i = \delta_4{}^i \Rightarrow T\sqrt{-g_{44}} = 1$)

$$\Delta S = \frac{\Delta M}{2} - \Delta \int_{x_4 = \text{const}} \frac{1}{T}\left(\frac{1}{2\kappa_0} R + F\right)\sqrt{{}^3 g}\, d^3x$$

According to (4.3) we have $\Delta S \geqslant 0$. Thus we get

$$\Delta \int_{x_4 = \text{const}} \frac{1}{T}\left(\frac{1}{2\kappa_0} R + F\right)\sqrt{{}^3 g}\, d^3x \leqslant 0$$

for a system with nonincreasing gravitational mass M. This result holds, e.g., for a system emitting gravitational radiation in transition from a frozen equilibrium state into thermodynamic equilibrium.

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[5] Ehlers has derived this equilibrium condition statistically.